THE TWO-EXPONENTIAL LIOUVILLE THEORY AND THE UNIQUENESS OF THE THREE-POINT FUNCTION

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Abstract

It is shown that in the two-exponential version of Liouville theory the coefficients of the three-point functions of vertex operators can be determined uniquely using the translational invariance of the path integral measure and the self-consistency of the two-point functions. The result agrees with that obtained using conformal bootstrap methods. Reflection symmetry and a previously conjectured relationship between the dimensional parameters of the theory and the overall scale are derived.

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Introduction: Although the quantisation of two-dimensional Liouville theory with a potential of the form $V_b = \mu e^{2b\tilde{\phi}(x)}$, where $\tilde{\phi}(x)$ is a scalar field and μ and b are constants, has been widely investigated [1], it still presents some problems. What is perhaps the most disturbing problem is the following: The three point functions of vertex operators $\exp[2\alpha_I\tilde{\phi}(x_I)]$ play a central role in the theory, in the sense that all N-point functions can be obtained from these by integration, and have the following form [2, 3]

$$\mathcal{G}_3 = \mathbf{C}_3 \prod_{I=1}^{3} |\frac{x_I - x_J}{L}|^{\Delta_{(IJ)}(\alpha)} \quad \text{where} \quad \mathbf{C}_3 = \Big(g_b(\xi) Z_0^b(\xi)\Big)\Big(\frac{K(\xi,\alpha_I)}{k'(-\xi)}\Big) \tag{1}$$

Here L is a constant scale, the $\Delta_{(IJ)}$ are known combinations of constant conformal weights, $\xi = q - \sum_{I=1}^{3}\alpha_I$, and $q = b + 1/b$, while $Z_0^b(\xi)$ is related to the zero-mode integration and can be computed explicitly. K and k are the functions defined in (26). The problem is that $\mathbf{C}_3$ can actually be computed only at the points $\xi = mb$ for $m \in Z_+$ and the extrapolation from these points to general values of ξ leaves a factor $g_b(\xi)$ in (1) undetermined. So far it has not been possible to determine this factor from first principles. Instead, what has been done is to make the so-called DOZZ Ansatz that

$$\mathbf{C}_3 = \frac{K(\xi,\alpha_I)}{k(-\xi)} \quad \Rightarrow \quad h_b(\xi) \equiv \; g_b Z_0^b \frac{k}{k'} \; = 1 \tag{2}$$

and check that the resulting three-point function satisfies some reasonable, but extra, conditions such as reflection symmetry and crossing symmetry. The latter check is the decisive one, but uses some special *four*-point functions [4].

In a previous paper [5] it was suggested that the gap in the direct derivation of $h(\xi)$ could be closed by using a potential of the form $V(\phi) = \mu_b e^{2b\tilde{\phi}} + \mu_c e^{2c\tilde{\phi}}$, where $bc = 1$, – this being the most general potential one can use in the path integral context for a conformal field theory of a single real scalar field without derivative interactions [6] – rather than the standard form V_b. As in the one-exponential theory, the $\mathbf{C}_3$ in the two-exponential theory can be calculated only at a discrete, but much larger, set of points, namely $\xi_{mn} = mb + nc$ and the equation corresponding to (1) takes the form

$$\mathbf{C}_3 = \Big(g(\xi) Z_0(\xi) \Big) \Big(\frac{K(\xi, \alpha_I)}{k'(-\xi)} \Big) \tag{3}$$

where $g(\xi)$ is *a priori* undetermined, the functions K and k are exactly the same as in (1) but Z_0 is not. It was shown in [5] that, subject to two conditions, the factor $g(\xi)$ could be fixed and thus the DOZZ Ansatz (2) could be *derived.* The conditions were (a) that the dimensional parameters μ_b and μ_c were related to the overall scale L of the system by an equation of the form $\Omega(\mu_b, \mu_c, L) = 1$, where Ω is the function defined in (25), and (b) that $h(\xi)$, defined as $h(\xi) = g(\xi) Z_0(\xi) k(-\xi)/k'(-\xi)$, had no singular points.

In the present paper we refine these results considerably and extend the analysis of the two-exponential theory. In particular we show that, subject to the mild technical condition given in (37), the DOZZ Ansatz follows from the translational invariance of the path integral measure and the self-consistency of the two-point functions. We also present an extrapolation of the fluctuating part of the path integral from the lattice points ξ_{mn}, for which the Z_0 part of the path integral is automatically

$$Z_0(\xi) = \frac{k'(-\xi)}{k(-\xi)} \tag{4}$$

This means that *the zero mode integral for the two-exponential theory produces exactly the factor k'/k that was postulated in the DOZZ Ansatz.*

As in [5], we use the path-integral formalism; and as the symmetries of the path-integral and the associated sum rules are of interest in their own right, a secondary purpose of the paper is to present these in a systematic way. The most important symmetries are those connected with the translational invariance of the measure and conformal (Weyl) invariance.

A by-product is an analysis of the two-point function $\mathcal{G}_2$. Although $\mathcal{G}_2$ cannot be defined directly because of conformal invariance, it can be defined both as a limit of $\mathcal{G}_3$

when one of the α's becomes zero, and as a volume integral of $\mathcal{G}_3$ when the corresponding α is b or c. The compatibility of the two definitions and the sum rule mentioned above lead to a simple linear homogeneous sum rule for the quantity $h(\xi)$. Together with the boundary conditions $h(\xi_{mn}) = \Omega^{m+n}$, obtained by direct computation, this sum rule fixes $h(\xi) = 1$ uniquely. It also fixes $\Omega = 1$ which is the relation between the dimensional constants μ_b and μ_c and the overall scale L that was conjectured in an earlier paper. The corresponding sum rule and boundary conditions in the one-exponential theory would imply only that $h(\xi)$ is periodic. If $\mathcal{G}_2$ is interpreted as an inner-product of states, then $h(\xi) = 1$ implies that for each conformal weight a zero-norm state decouples, so that there is only one physical state. The decoupling is equivalent to reflection invariance, which therefore emerges as an output.

The generating functional: The generating functional of the two-exponential Liouville theory is defined as

$$Z[J] = \int [d\tilde{\phi}]\, e^{-\int d^2x \sqrt{g(x)}\ \left[\frac{1}{4\pi}\tilde{\phi}\Delta\tilde{\phi} + \frac{q}{4\pi}\mathcal{R}\tilde{\phi} + \mu_b e^{2b\tilde{\phi}} + \mu_c e^{2c\tilde{\phi}} - J(x)\tilde{\phi}(x)\right]} \tag{5}$$

where Δ is the Laplace-Beltrami operator, $\mathcal{R}$ is the Ricci scalar, the coupling constants μ_b, μ_c have dimensions of mass squared, and b and c are dimensionless constants. For the N-point functions of vertex operators we have $e^{\int d^2x\sqrt{g}J\tilde{\phi}} = \prod_{I=1}^{N} \nu_I \sqrt{g} e^{2\alpha_I\tilde{\phi}}$, but we need not yet specialise to this case. The ν_I are constants which, like the μ_b and μ_c, have to be renormalised because of the short distance singularities of the Green's function defined by the equation $\Delta G(x,y) = \frac{\pi}{\sqrt{g}}\delta^2(x-y)$. As explained in detail in [5], the Green's function for the non-coincident and coincident arguments is defined as:

$$G(x,y) = -\frac{1}{2}\ln\frac{|\, x-y\,|}{L} + \mathcal{O}(V^{-1}) \quad \text{and} \quad G(x,x+dx) \equiv -\frac{1}{2}\ln[\frac{ds}{L}] + \frac{1}{4}\ln\sqrt{g(x)} \tag{6}$$

respectively, where ds is the infinitesimal geodesic separation, L sets the overall scale, and V is the volume of the space.

As the Green's function $G(x,x)$ occurs only in the form $-\alpha^2 G(x,x)$ at each vertex, its infinite part may be absorbed by renormalising μ_b, μ_c and ν_I. Let us define the renormalisations of μ_b, μ_c and ν_I to be

$$\mu_b \to \mu_b(\Lambda\frac{ds}{L})^{2(b^2-qb)}, \quad \mu_c \to \mu_c(\Lambda\frac{ds}{L})^{2(c^2-qc)}, \quad \nu_I \to \nu_I(\Lambda\frac{ds}{L})^{2(\alpha_I^2-q\alpha_I)} \tag{7}$$

where Λ is the dimensionless ultra-violet renormalisation scale. Anticipating the result that $b+c=q, \quad bc=1$ from Weyl invariance, it is easy to see that μ_b and μ_c have the naive scaling dimensions with respect to Λ. [1] Clearly the b^2, c^2, α^2 powers of ds in the above equation cancel the infinities coming from the Green's functions at coincident points. Counting the powers of the remaining factors of ds and Λ, we find, after a little algebra, that the renormalised functional integral is proportional to

$$[\frac{ds}{L}]^{-2q^2}\prod_{I=1}^{N}\Lambda^{2(\alpha_I^2-q\alpha_I)} \tag{8}$$

Since the factor $[ds/L]^{-q^2}$ may be absorbed in the Polyakov conformal anomaly term which is proportional to q^2, we see that the Λ dependence of the renormalised functional integral cancels except for the contribution coming from the external sources.

The conformal weights Δ_α of the fields $\exp\,[2\alpha_I\tilde{\phi}(x_I)]$ may be read off directly by varying the path integral with respect $\sqrt{g(x_I)}$. Taking the $\sqrt{g}$ dependence of $\alpha_I^2 G(x_I,x_I)$ and the cross-term $q\alpha_I\int d^2x\int d^2y\sqrt{g(x)}\mathcal{R}(x)G(x,y)\delta(y-x_I)$ in the Gaussian integration into account, this yields $\Delta_\alpha=\alpha(q-\alpha)$. From (7) it is clear that the scaling dimensions of ν_I are minus the conformal weight of corresponding field. Thus fields of the same conformal weight are renormalised in the same way.

[1] There is arbitrariness in the definition of renormalisation associated with the translational invariance of the path integral measure. However, this arbitrariness does not affect equation (8).

A sum rule for the generating functional: If we now specialise to the case where the external current is of the form $\prod_{I=1}^N \nu_I e^{\int d^2x\sqrt{g}J\tilde{\phi}} = \prod_{I=1}^N \nu_I \sqrt{g} e^{2\alpha_I\tilde{\phi}}$, the renormalised functional integral is defined by

$$Z[J] = \int [d\tilde{\phi}]\; e^{-\int d^2x\sqrt{g(x)}\;\left[\frac{1}{4\pi}\tilde{\phi}\Delta\tilde{\phi}+\frac{q}{4\pi}\mathcal{R}\tilde{\phi}+\sqrt{g}^{b^2}\;\mu_b e^{2b\tilde{\phi}}+\sqrt{g}^{c^2}\;\mu_c e^{2c\tilde{\phi}}\right]} \prod_{I=1}^{N} \sqrt{g}^{\alpha_I^2}\nu_I e^{2\alpha_I\tilde{\phi}(x_I)} \tag{9}$$

where we are using the renormalisation prescription in (7) and we have absorbed, for convenience, the $\ln\sqrt{g}$ part of $G(x,x)$ by letting $e^{2\alpha\phi} \to \sqrt{g}^{\alpha^2} e^{2\alpha\phi}$. It is then understood that $G_R(x,x) = 0$.

The translation invariance of the path integral measure can now be formulated as follows:

$$\int [d\tilde{\phi}]\; \frac{\delta}{\delta\tilde{\phi}(x)} \left[e^{-S[\tilde{\phi}]+\int d^2x\sqrt{g}J(x)\tilde{\phi}(x)}\right] = 0 \tag{10}$$

where $\frac{\delta}{\delta\tilde{\phi}(x)}$ is the generator of translations on the space of fields. From (10) we derive the quantum equations of motion for $Z[J]$:

$$\frac{1}{4\pi}\Delta\frac{\delta Z[J]}{\delta J(x)} = -\Big(b\mu_b\sqrt{g}^{b^2} Z[J_{b,x}] + c\mu_c\sqrt{g}^{c^2} Z[J_{c,x}]\Big) + \frac{1}{2}\big(J(x) - \frac{q}{4\pi}\mathcal{R}(x)\big)Z[J] \tag{11}$$

where

$$J_{b,x}(y) = J(y) + 2b\frac{1}{\sqrt{g(y)}}\delta^2(y-x), \quad J_{c,x}(y) = J(y) + 2c\frac{1}{\sqrt{g(y)}}\delta^2(y-x) \tag{12}$$

Carrying out the same procedure for the constant, zero-mode, part ϕ_0 of $\tilde{\phi}$, we obtain the integrated form of (11), namely

$$\int d^2x\sqrt{g}\Big(b\,\mu_b\sqrt{g}^{b^2} Z[J_{b,x}] + c\,\mu_c\sqrt{g}^{c^2} Z[J_{c,x}]\Big) = \frac{1}{2}\left(J_0 - q\chi\right) Z[J] \tag{13}$$

where

$$J_0 = \int d^2x\sqrt{g}J(x) \quad \text{and} \quad \chi = \frac{1}{4\pi}\int d^2x\sqrt{g}\mathcal{R}(x) \tag{14}$$

χ being the Euler number of the underlying manifold. Eq. (13) embodies the sum rule for the generating functional.

Weyl transformations: A local Weyl transformation can be performed by varying the generating functional with respect to $\sqrt{g(x)}$ where $x \neq x_I$, the external points. For $Z[J]$ in (9), we find

$$\frac{\delta Z}{\delta\sqrt{g}} = (1+b^2)\sqrt{g}^{b^2}\mu_b\, Z[J_{b,x}] + (1+c^2)\sqrt{g}^{c^2}\mu_c\, Z[J_{c,x}] + \frac{q}{4\pi}\Delta\frac{\delta Z[J]}{\delta J(x)} \tag{15}$$

The third term may be eliminated using (11) to get

$$\Big(1+b^2-qb\Big)\sqrt{g}^{b^2}\mu_b Z[J_{b,x}]+\Big(1+c^2-qc\Big)\sqrt{g}^{c^2}\mu_c Z[J_{c,x}]-\Big(\frac{q}{4\pi}\mathcal{R}(x)-J(x)\Big)\frac{q}{2}Z(J) \tag{16}$$

The Weyl condition is that the variation $\frac{\delta Z}{\delta\sqrt{g}}$ should be proportional to the external current namely $\frac{q}{4\pi}\mathcal{R}-J$. Since this has to be valid for all currents J, the condition for Weyl invariance is that the first two terms must vanish and we have

$$q=(b+c) \quad \text{and} \quad bc=1, \tag{17}$$

The above approach may be contrasted with the one in [5] where (17) was derived only after the path integral was evaluated.

The N-point functions: For the computation of general N-point functions of vertex operators, we let the underlying manifold be a two dimensional sphere. In that case, $\chi=2$ and there is only one zero-mode for $\tilde{\phi}$, namely the constant ϕ_0. As explained in detail in [5], the expression for the N-point function may be simplified by using a Sommerfeld-Watson transform [7] for the exponential of an integrated vertex operator. With $\tilde{\phi}=\phi_0+\phi$ the resulting expression for the N-point function takes the form

$$\mathcal{G}_N = \int d\phi_0 \int \frac{du\,dv}{\Gamma(1+iu)\Gamma(1+iv)}\,\frac{e^{2(i(bu+cv)-\xi_N)\phi_0}}{\sinh\pi u\,\sinh\pi v}\times \mathbf{C}_N(iu,iv) \tag{18}$$

where

$$\mathbf{C}_N(iu,iv) = \int d\phi\, U_b^{iu}\, U_c^{iv}\, e^{-\int d^2x\sqrt{g}\left[\frac{1}{4\pi}\phi\Delta\phi+\frac{q}{4\pi}R\phi\right]}\mathbf{\Pi}_N \quad \text{with} \quad \xi_N = q-\sum_{I=1}^{N}\alpha_I \tag{19}$$

and

$$\mathbf{\Pi}_N = \prod_{I=1}^{N} \sqrt{g}^{\alpha_I^2} \psi_I e^{2\alpha_I \tilde{\phi}(x_I)}, \qquad U_b(\phi) = \mu_b \int d^2x (\sqrt{g})^{qb} e^{2b\phi} \tag{20}$$

and similarly for $b \leftrightarrow c$. The integral $\mathbf{C}_N(iu, iv)$ is a Gaussian path integral for the fluctuations ϕ which, for $iu = m$ and $iv = n$, m and n being positive integers, can be done in a straightforward manner and produces an ordinary multiple integral.

In terms of the N-point functions $\mathcal{G}_N(x_I, \alpha_I)$, the sum rule (13) takes the form

$$\int d^2x \sqrt{g}\, [b\mu_b \mathcal{G}_{N+1}(x_I, x, \alpha_I, b) + c\mu_c \mathcal{G}_{N+1}(x_I, x, \alpha_I, c)] = -\xi_N \mathcal{G}_N(x_I, \alpha_I) \tag{21}$$

and thus relates the (integrated) $N+1$-point function to the N-point function. Note that the above equation requires that $\xi_N \neq 0$ because $\mathcal{G}_N \to \infty$ as $\xi_N \to 0$, making the right hand side indefinite.

The Three-point function: As is well-known, the three-point function is the lowest N-point function for which conformal invariance does not require the extraction of an infinite group volume factor. If we choose ξ to be pure imaginary and integrate over the zero-mode ϕ_0 in (18) we obtain a delta function $\delta(\xi - bu - cv)$, in which case the coefficient $\mathbf{C}_3$ may be written as

$$\mathbf{C}_3(\xi, i(u+v)) = \int d\phi e^{-\frac{1}{4\pi}\int d^2x\sqrt{g}[\phi\Delta\phi + qR\phi]} U_b^{iu} U_c^{iv} \mathbf{\Pi_3} \tag{22}$$

Apart from the spectator variables $\alpha_I - \alpha_J$, we see that, due to the delta-function, $\mathbf{C}_3$ is a function of only two variables, chosen as ξ and $u+v$ for convenience. This is the great advantage of using the Sommerfeld-Watson transform. We then have in the infinite volume limit,

$$\mathcal{G}_3 = \int \frac{dudv}{\Gamma(1+iu)\Gamma(1+iv)} \frac{\delta(\xi - ibu - icv)}{\sinh\pi u \, \sinh\pi v} \mathbf{C}_3(\xi, i(u+v)) \tag{23}$$

The problem is that $\mathbf{C}_3$ can only be computed at the points $u, v = -im, -in$ for $m, n \in Z_+$, where, as shown in [5], it is given by, for $\xi_{mn} \equiv bm + cn$,

$$\mathbf{C}_3(\xi_{mn}, m+n) = (-1)^{m+n} m! n! \lambda^{\xi_{mn}} \Omega^{m+n} \Big(\frac{K(\xi_{mn}, \alpha_I)}{k'(-\xi_{mn})}\Big) \prod_{I=1}^{3} \mid \frac{x_{IJ}}{L} \mid^{\Delta_{IJ}(\alpha)} \tag{24}$$

where

$$\lambda = \left(\frac{\mu_b \Phi_b}{\mu_c \Phi_c}\right)^{\frac{1}{b-c}}, \quad \Omega^{c-b} = \frac{(\mu_b L^2 \Phi_b)^c}{(\mu_c L^2 \Phi_c)^b}, \quad \Phi_b = \pi\gamma(b^2)(b^2)^{2-qb}, \quad \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \quad (25)$$

The functions $k(\xi)$ and $K(\xi, \alpha_I)$ are defined by the following equations:

$$\ln k(\xi) = \int_0^\infty \frac{dt}{t}\Big((\frac{q}{2} - x)^2 e^{-2t} - \frac{\sinh^2(\frac{q}{2} - x)t}{\sinh bt\ \sinh ct}\Big) \text{ and } K(\xi, \alpha_I) = k'(0)\prod_{I=1}^{3} \frac{k(2\alpha_I)}{k(\xi + 2\alpha_I)} \quad (26)$$

To extrapolate $\mathbf{C}_3$ to other values of ξ we note that although the arguments of k's are constrained by the relation $\sum_I \alpha_I = q - \xi_{mn}$, they range over the whole real axis for fixed ξ_{mn}. Hence the only reasonable extrapolation is the obvious one, $k(\xi_{mn}, \alpha_I) \to k(\xi, \alpha_I)$, in which case $K(\xi_{mn}, \alpha_I) \to K(\xi, \alpha_I)$. Unfortunately, this argument is not valid for the rest of $\mathbf{C}_3(\xi_{mn}, m{+}n)$, which depends only on ξ_{mn}. Thus the most general extrapolation of (24) is

$$\mathbf{C}_3(iu, iv, \xi) = \cosh\pi u \cosh\pi v \Gamma(1{+}iu)\Gamma(1{+}iv)\lambda^\xi \Omega^{i(u+v)} \frac{K(\xi, \alpha_I)}{k'(-\xi)} f(\xi, u{+}v) \mid \frac{x_{IJ}}{L} \mid^{\Delta_{(IJ)}} \quad (27)$$

where the cosh terms take care of the $(-1)^{m+n}$ terms and $f(\xi, u + v)$ is an arbitary function with $f(\xi = mb + nc, m + n) = 1$. Then $\mathcal{G}_3$ becomes

$$\mathcal{G}_3 = \lambda^\xi h(\xi) \frac{K(\xi, \alpha_I)}{k(-\xi)} \prod_{I=1}^{3} \mid \frac{x_{IJ}}{L} \mid^{\Delta_{(IJ)}(\alpha_I)} \quad (28)$$

where

$$h(\xi) = \frac{k(-\xi)}{k'(-\xi)} Z_0(\xi) \quad \text{and} \quad Z_0(\xi) = \int \frac{du dv\ \delta(\xi - ibu - icv)}{\tanh\pi u\ \tanh\pi v} \Omega^{i(u+v)} f[\xi, i(u{+}v)] \quad (29)$$

Since the function f is unknown, we cannot proceed from (29). Hence we take an alternative route using the two-point function.

Uniqueness (An Application of the Sum Rule): For two and three point functions the sum rule (21) in the infinite volume limit is

$$b\mu_b \int d^2x \mathcal{G}_3(x_I, x, \alpha_I, b) + c\mu_c \int d^2x \mathcal{G}_3(x_I, x, \alpha_I, c) = -\xi_2 \mathcal{G}_2(x_I, \alpha_I), \qquad \xi_2 \neq 0 \quad (30)$$

This is not useful unless we have an alternative definition for the two-point function. Such a definition may be obtained by regarding it as twice [2] the limit of the three-point function as $\alpha_3 \to 0$ and $x_3 \to \infty$. It is easy to see that the limit is non-zero only if $\Delta_1 = \Delta_2$ *i.e* if $\alpha_1 = \alpha_2$ or $\alpha_1 = q - \alpha_2$. Since for scattering states $\alpha_I = \frac{q}{2} + i\beta_I$, the quantities $\alpha_1 - \alpha_2$ and $\xi_2 = q - \alpha_1 - \alpha_2$ are pure imaginary, and we obtain

$$\mathcal{G}_2(\alpha_1, \alpha_2; x_{12}) = 4\pi\lambda^{\xi_2} \mid \frac{x_{12}}{L} \mid^{-(\Delta_1+\Delta_2)} h(\xi_2)R(\xi_2)[\delta(\beta_1 - \beta_2) + \delta(\beta_1 + \beta_2)] \tag{31}$$

where $R(\xi) = k(\xi)/k(-\xi)$.

To compute the left hand side of (30) we note that the first integral is

$$\lambda^{\xi_2 - b} h(\xi_2 - b) \frac{K(\alpha_1, \alpha_2, b)}{k(-\xi_2 + b)} |\frac{x_{12}}{L}|^{2(\Delta_b - \Delta_1 - \Delta_2)} \times \zeta \tag{32}$$

where

$$\zeta = \int d^2x_3 |\frac{x_{31}}{L}|^{-2(\Delta_b + D)} |\frac{x_{23}}{L}|^{-2(\Delta_b - D)} = 2\pi^2 \mid \frac{x_{12}}{L} \mid^{-2\Delta_b} \delta(D) \tag{33}$$

and $\Delta_b = 1$ and $D = \Delta_1 - \Delta_2 = \beta_1^2 - \beta_2^2$. The delta functions in these equations and the condition $\xi_2 \neq 0$ in (30) mean that we only need the coefficient for $\alpha_1 = \alpha_2 \equiv \alpha$, which is easily computed to be

$$\frac{K(\alpha, \alpha, b)}{k(-\xi_2 + b)} = -\xi_2^2 k'(0) R(\xi_2) \frac{k(2b)}{k^2(b)} = \xi_2^2 R(\xi_2) \frac{\Phi_b}{\pi b}, \qquad \xi_2 = q - 2\alpha \tag{34}$$

Inserting these formulae into the sum rule (30) and using the identity $\lambda^b = \mu_b \Phi_b \Omega$, and similarly for c, we get the sum rule

$$h(\xi_2 + b) + h(\xi_2 + c) = 2\Omega h(\xi_2) \quad \text{with} \quad h(mb + nc) = \Omega^{m+n} \tag{35}$$

for the unknown function $h(\xi_2)$. The second equation is obtained by explicit computation from (24). The question is whether the equations in (35) determine $h(\xi_2)$ uniquely.

[2] Actually any constant independent of ξ is allowed *a priori*, but the requirement that the sum rule be satisfied at the points $\xi = mb + nc$ fixes the constant to be two.

We show that they do subject to a mild technical assumption to be introduced presently. First let us restrict ourselves to the case when b is fractional i.e. $b = r/s$ where $r > s$ are positive integers with no common factor and lowest common multiple rs. Next we rewrite the sum rule (35) in terms of new variables defined as follows:

$$y \equiv e^{\xi} \quad \Rightarrow \quad h(e^{-\frac{r}{s}}y) + h(e^{-\frac{s}{r}}y) = 2\Omega h(y) \tag{36}$$

We now make the assumption[3]

$$\lim_{y \to 0} \frac{h\big(e^{-(\frac{r}{s})^{\pm 1}}y\big)}{h(y)} \;\equiv\; C_{\pm} < \infty \tag{37}$$

Considering the restriction

$$y_t = e^{\frac{t}{rs}} \quad \Rightarrow \quad h(e^{\frac{t-r^2}{rs}}) + h(e^{\frac{t-s^2}{rs}}) = 2\Omega h(e^{\frac{t}{rs}}), \quad \text{where} \quad t \in Z_+ \tag{38}$$

we see that the symmetry group we need to implement is the dilatation group. Thus we may expand a solution of (38) as follows:

$$h(y_t) = \sum_{p=0}^{N} C_p y_t^{\sigma_p} \quad \text{where} \quad e^{-r^2\omega\sigma_p} + e^{-s^2\omega\sigma_p} - 2\Omega = 0 \quad \text{and} \quad \omega = \frac{1}{rs} \tag{39}$$

N $(0 \leq N \leq r^2)$ being a finite number. This follows because the sum rule restricts the number of C_ps to be the dimension of the solution space of the polynomial equation in (39). This is of course true only on the first sheet of the covering of the y variable. In the general case we would also have a sum over the infinite number of coverings.

$$h(y) = \sum_{p=0}^{N} \sum_{n=-\infty}^{\infty} C_{p_n} y^{\sigma_{p_n}} \quad \text{where} \quad \sigma_{p_n} = \sigma + 2\pi i n rs \tag{40}$$

We now use the second equation in (35) to write, for the special values $\xi = mb + nc \Rightarrow t = mr^2 + ns^2$,

$$\sum_{p=0}^{N} C_p y_t^{\sigma_p} = \Omega^{m+n} \Rightarrow C_N = 1, \quad \sigma_N = 0, \quad C_{p \neq N} = 0 \Rightarrow \Omega = 1 \tag{41}$$

[3] It is probable that this could be actually proved directly from the sum rule (35) which rules out a whole class of functions.

The latter follows as the only consistent solution since the finite sum on the left hand side cannot in general match the right hand side which can be made arbitrarily large by letting m and n tend to infinity. Note that the covering does not play a role in the discussion of (41). It then follows that

$$h(\frac{t}{rs}) = 1 \quad \forall \ \ t \in Z_+ \tag{42}$$

Thus for $b = r/s$ the function $h(\xi)$ is unity when restricted to the subset of points $\xi = t/rs$.

This is as far as we can go for a fixed $b = r/s$. However we now invoke the fact that (29) implies that $h(\xi)$ is continuous in b and ξ. We then define $\tilde{b} = \tilde{r}/\tilde{s}$ where $\tilde{r} = lr + 1$ and $\tilde{s} = ls$. It follows that $\tilde{r}$ and $\tilde{s}$ have no common factor, $\tilde{b} - b = 1/ls$ and, by applying the above result to the tilde-variables, $h(t/l^2 sr(1 + 1/lr) = 1$. But this means that in any $1/l$ neighbourhood of b there is a $\tilde{b}$ for which $h(\xi) = 1$ at points which are separated by distances of order $1/l^2$. As these distances tend to zero as l tends to infinity, we see that this is compatible with the continuity of $h(\xi)$ in b and ξ only if $h(\xi) = 1$ for all ξ.
Reflection symmetry: Once $h(\xi) = 1$ it follows that the denominator in the three-point function is invariant under the reflection $\alpha_I \to q - \alpha_I$ for each I and thus the three-point function is covariant with respect to reflection symmetry in the sense that

$$\mathcal{G}_3(q - \alpha_1, \alpha_2, \alpha_3) = R(q - 2\alpha_1)\mathcal{G}_3(\alpha_1, \alpha_2, \alpha_3) \tag{43}$$

where the prefactor depends only on the reflected parameter α_1. Thus in the two-exponential theory, reflection covariance is an output rather than an input.

It is interesting to note how this reflection covariance expresses itself in terms of the two-point function defined. If we interpret the two-point function normalised by the volume factor 4π, as the inner product of primary states

$$< \alpha_1, \alpha_2 > \equiv \lim_{x \to 0} \frac{1}{4\pi} \mid x \mid^{2\Delta_\alpha} \mathcal{G}_2(q - \alpha_1, \alpha_2; x) \tag{44}$$

we have

$$\begin{pmatrix} <\alpha,\alpha> & <\alpha,q-\alpha> \\ <q-\alpha,\alpha> & <q-\alpha,q-\alpha> \end{pmatrix} = \begin{pmatrix} 1 & R^{-1}(q-2\alpha) \\ R(q-2\alpha) & 1 \end{pmatrix}\delta(0) \tag{45}$$

It is clear that the matrix in (45) is hermitian and has zero determinant. Hence one linear combination of the states, namely $|\alpha> -R(q-2\alpha)|q-\alpha>$, has zero norm and decouples. Thus effectively,

$$|\alpha> \;=\; R(q-2\alpha)|q-\alpha> \tag{46}$$

which means that there is actually only one physical state for each conformal weight Δ_α. This may seem surprising but from (43) it is seen to be a manifestation of the reflection covariance.

Comparison of the one and two-exponential path integrals: In order to compare the one and two exponential theories, we begin by recalling that $\mathbf{C}_3$ in both the theories is defined in terms of the correlation functions of vertex operators in a free field theory. In the two-exponential theory, the relevant integral is given by

$$\int d\phi e^{-S(\phi)} U_b^{iu} U_c^{iv} \mathbf{\Pi}_3 = \Gamma(1+iu)\Gamma(1+iv)\frac{K(-\xi,\alpha_I)}{k'(-\xi)} f[\xi, i(u+v)] \tag{47}$$

where $S(\phi)$ is the Action for the free theory. The corresponding equation, in the one-exponential theory, is obtained by letting $v \to 0$ and takes the form

$$\int d\phi e^{-S(\phi)} U_b^{iu} \mathbf{\Pi}_3 = \Gamma(1+iu)\frac{K(-\xi,\alpha_I)}{k'(-\xi)} \times f[\xi, iu] \tag{48}$$

The integral corresponding to the zero-mode integral in (29) in the one-exponential theory for an *a priori* arbitrary $f[\xi, iu]$ can be performed to yield $f[\xi, c\xi]/\tanh\pi c\xi$. If we *assume* that the one-exponential theory leads to the DOZZ Ansatz, then the function $f[\xi, iu]$ is determined *uniquely* to be $b\tan(\pi u)k'(-ibu)/k(-ibu)$ for $\xi = ibu$. We may now ask, what choice, if any, for the function $f[\xi, i(u+v)]$ in the two-exponential theory will produce the DOZZ Ansatz. It is easy to see that if we choose

$$f[\xi, i(u+v)] = \frac{k'(-\xi)}{k(-\xi)}[b\tanh\pi u + c\tanh\pi v] \tag{49}$$

the Z_0 integral in (29) produces k'/k in accordance with the DOZZ Ansatz. This choice has the virtue that it reduces to the one-exponential result in the limit $v \to 0$. However, one can easily convince oneself that there exist other choices for the function $f[\xi, i(u+v)]$. If we choose, for example, $f[\xi, i(u+v)] = \mathrm{sgn}[i(u+v)]$ (the sgn-factor is necessary to preserve the symmetry of the path integral under a change of sign of b, c and the α's) and integrate over u and v first, we obtain,

$$Z_0(\xi) = \int_0^\infty dt \frac{\sinh((q-2\xi)t)}{\sinh(bt)\sinh(ct)}, \qquad t \equiv |\phi_0| \tag{50}$$

Up to a regulating term, [4] this will be recognized, from the definition of $k(\xi)$ in (26), as $k'(-\xi)/k(-\xi)$. Since the choice for the extrapolation function is not unique we conclude that any result that is based on an extrapolation is not conclusive. It is therefore desirable to have a direct proof of the DOZZ result for the three-point function without any reference to the extrapolation. This is exactly what was achieved in this paper by showing that the sum rule in the two-exponential theory has a unique (constant) solution. In contrast, in the one-exponential theory, if the specific extrapolation leading to the DOZZ Ansatz is not made, the final result can only be obtained up to a periodic function of ξ.

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[4] The regulating term can be obtained naturally by modifying the path-integral measure $d\phi_0$ to $\lim_{\epsilon\to 0} d\phi_0 \big[1 - \exp(-|\phi_0|/\epsilon)\big]$.

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